Sec. 10.6 Constant Coefficient Homogeneous Systems (part 3)

Complex eigenvalues

Suppose that A is an $n \times n$ matrix with real entries. Suppose that A has a complex eigenvalue

$$\lambda = \alpha + i\beta$$

and let the associated eigenvector be

$$\vec{x} = \vec{u} + i\vec{v}$$

with real vector components $\vec{u}$ and $\vec{v}$. Then the two linearly independent solutions of the homogeneous system $\vec{y}' = A\vec{y}$ are the real and imaginary parts of

$$\boxed{e^{\alpha t}\left(\cos\beta t + i\sin\beta t\right)\left(\vec{u} + i\vec{v}\right)}$$

that is,

$$\vec{y}_1 = e^{\alpha t}\left(\vec{u}\cos\beta t - \vec{v}\sin\beta t\right) \quad \text{and} \quad \vec{y}_2 = e^{\alpha t}\left(\vec{u}\sin\beta t + \vec{v}\cos\beta t\right)$$

Ex. Find the general solution of the system $\vec{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix}\vec{y}$.

Answer: First, we find the eigenvalues.

$$A\vec{x} = \lambda\vec{x} \iff (A - \lambda I)\vec{x} = \vec{0}$$

$$\iff \det(A - \lambda I) = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & -5 \\ 5 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -(\lambda-4) & -5 \\ 5 & -(\lambda+2) \end{vmatrix} = 0$$

$$\iff (\lambda-4)(\lambda+2) + 25 = 0$$

$$\iff \lambda^2 - 2\lambda - 8 + 25 = 0$$

$$\iff (\lambda-1)^2 + 16 = 0$$

$$\Rightarrow \lambda = 1 \pm 4i \quad \Rightarrow \quad \begin{matrix} \alpha = 1 \\ \beta = 4 \end{matrix}$$

The two eigenvalues form a complex-conjugate pair.

First, we find the eigenvector corresponding to one of the eigenvalues, for example, $\lambda = 1+4i$.

$$(A-\lambda I)\vec{x} = \vec{0} \iff \begin{bmatrix} 4-(1+4i) & -5 \\ 5 & -2-(1+4i) \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff \begin{bmatrix} 3-4i & -5 \\ 5 & -3-4i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

There are different ways to solve for x_1 and x_2, one possibility would be to use the augmented matrix:

$$\left[\begin{array}{cc|c} 3-4i & -5 & 0 \\ 5 & -3-4i & 0 \end{array}\right] \overset{R_1 \leftrightarrow R_2}{\iff} \left[\begin{array}{cc|c} 5 & -3-4i & 0 \\ 3-4i & -5 & 0 \end{array}\right]$$

$$\overset{R_2 \to R_2(3+4i)}{\iff} \left[\begin{array}{cc|c} 5 & -3-4i & 0 \\ 25 & -5(3+4i) & 0 \end{array}\right]$$

$$\overset{R_2 \to R_2 - 5R_1}{\iff} \left[\begin{array}{cc|c} 5 & -3-4i & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\overset{R_1 \to \frac{R_1}{5}}{\iff} \left[\begin{array}{cc|c} 1 & -\frac{3+4i}{5} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus,

$$x_1 - \frac{3+4i}{5} \cdot x_2 = 0 \implies x_1 = \frac{3+4i}{5} \cdot x_2$$

There are infinitely many ways to get a non-zero eigenvector. A simple choice is $x_2 = 5$ so $x_1 = 3+4i$, so

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4i \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i\begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

The real and imaginary parts of this vector are $\vec{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$. The two solution functions are the real and imaginary parts of:

$$f = e^{\alpha t}(\cos\beta t + i\sin\beta t)(\vec{u} + i\vec{v})$$

$$f = e^{t}(\cos 4t + i\sin 4t)\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} + i\begin{bmatrix} 4 \\ 0 \end{bmatrix}\right)$$

$$f = e^{t}\left(\begin{bmatrix} 3 \\ 5 \end{bmatrix}\cos 4t - \begin{bmatrix} 4 \\ 0 \end{bmatrix}\sin 4t\right) + i \cdot e^{t}\left(\begin{bmatrix} 4 \\ 0 \end{bmatrix}\cos 4t + \begin{bmatrix} 3 \\ 5 \end{bmatrix}\sin 4t\right)$$

Thus,

$$\vec{y}_1 = Re\{f\} = e^t \cdot \left(\begin{bmatrix} 3 \\ 5 \end{bmatrix} \cos 4t - \begin{bmatrix} 4 \\ 0 \end{bmatrix} \sin 4t \right) = e^t \cdot \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix}$$

$$\vec{y}_2 = Im\{f\} = e^t \cdot \left(\begin{bmatrix} 4 \\ 0 \end{bmatrix} \cos 4t + \begin{bmatrix} 3 \\ 5 \end{bmatrix} \sin 4t \right) = e^t \cdot \begin{bmatrix} 4\cos 4t + 3\sin 4t \\ 5\sin 4t \end{bmatrix}$$

The general solution is a linear combination of the two solution functions:

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2$$

$$\vec{y} = c_1 e^t \begin{bmatrix} 3\cos 4t - 4\sin 4t \\ 5\cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 4\cos 4t + 3\sin 4t \\ 5\sin 4t \end{bmatrix}$$

$$\boxed{\vec{y} = \begin{bmatrix} e^t(3c_1 + 4c_2)\cos 4t + e^t(-4c_1 + 3c_2)\sin 4t \\ 5c_1 e^t \cdot \cos 4t + 5c_2 e^t \sin 4t \end{bmatrix}}$$

References: Trench, William F. (2013). *Elementary Differential Equations.* Retrieved from Digital Commons: http://digitalcommons.trinity.edu/mono/8