Sec. 10.5 Constant Coefficient Homogeneous Systems (part 2)

Repeated eigenvalues

Suppose that the $n \times n$ matrix A has an eigenvalue λ of multiplicity 2 or higher and the associated eigenspace has dimension 1, so all associated eigenvectors are scalar multiples of an eigenvector $\vec{v}$. Then

$$\vec{y}_1 = \vec{v}\,e^{\lambda t} \quad ; \quad \vec{y}_2 = \vec{v}\,t\,e^{\lambda t} + \vec{w}\,e^{\lambda t}$$

are linearly independent solutions of $\vec{y}\,' = A\vec{y}$, where

- The eigenpair $(\lambda, \vec{v})$ is a solution to

$$A\vec{v} = \lambda\vec{v} \quad \Leftrightarrow \quad (A - \lambda I)\vec{v} = \vec{0}$$

- The generalized vector $\vec{w}$ is obtained as a solution to the following matrix equation:

$$(A - \lambda I)\vec{w} = \vec{v}$$

Ex. Find the general solution of the system $\vec{y}\,' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \vec{y}$.

Answer: Set up the eigenvalue problem:

$$A\vec{v} = \lambda\vec{v}$$

$$\Leftrightarrow \quad (A - \lambda I)\vec{v} = \vec{0}$$

$$\Rightarrow \quad \det(A - \lambda I) = 0$$

Compute the eigenvalues:

$$\begin{vmatrix} 11-\lambda & -25 \\ 4 & -9-\lambda \end{vmatrix} = 0 \quad \Leftrightarrow \quad \begin{vmatrix} -(\lambda-11) & -25 \\ 4 & -(\lambda+9) \end{vmatrix} = 0$$

$$\Leftrightarrow \quad (\lambda-11)(\lambda+9) + 100 = 0$$

$$\Leftrightarrow \quad \lambda^2 - 2\lambda - 99 + 100 = 0$$

$$\Leftrightarrow \quad \lambda^2 - 2\lambda + 1 = 0$$

$$\Leftrightarrow \quad (\lambda - 1)^2 = 0$$

$$\Rightarrow \quad \lambda_1 = \lambda_2 = 1$$

We have a repeated eigenvalue.

The first eigenvector is obtained from:

$$(A - \lambda I)\vec{v} = \vec{0}$$

$$\begin{pmatrix} 11-\lambda & -25 \\ 4 & -9-\lambda \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 10 & -25 \\ 4 & -10 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$10x_1 - 25x_2 = 0$$
$$4x_1 - 10x_2 = 0$$

Dividing the first equation by 5 and the second by 2 we obtain:

$$2x_1 - 5x_2 = 0$$
$$2x_1 - 5x_2 = 0$$

Hence,

$$x_1 = \frac{5}{2}x_2$$

There are infinitely many choices (as long as we obtain a non-zero eigenvector). For simplicity, pick $x_2 = 2$, then $x_1 = 5$.

$$\vec{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

so

$$\vec{y}_1 = \vec{v} \cdot e^{\lambda t} = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^{t}$$

The second solution is obtained as:

$$\vec{y}_2 = \vec{v}te^{\lambda t} + \vec{w}e^{t}$$

where

$$(A - \lambda I)\vec{w} = \vec{v}$$

$$\begin{bmatrix} 11-1 & -25 \\ 4 & -9-1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 10 & -25 \\ 4 & -10 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$$

$$10x_1 - 25x_2 = 5$$
$$4x_1 - 10x_2 = 2$$

After dividing the 1st equation by 5 and the 2nd equation by 2, we obtain

$$2x_1 - 5x_2 = 1$$

Thus,

$$x_1 = \frac{1+5x_2}{2}$$

The simplest choice is to pick $x_2 = 0$ (a zero value is allowed since this time it does *not* result in a zero eigenvector).

$$\vec{w} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$$

so

$$\vec{y}_2 = \vec{v}\,te^t + \vec{w}e^t = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^t$$

$$\boxed{\vec{y}_2 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2}}$$

Finally,

$$\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2$$

$$\vec{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + t \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t \right)$$

References: Trench, William F. (2013). *Elementary Differential Equations.* Retrieved from Digital Commons: http://digitalcommons.trinity.edu/mono/8