Sec. 10.4 Constant Coefficient Homogeneous Systems

We seek a solution to the homogeneous system of *n* first order linear equations with constant coefficients:

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \ldots + a_{1n}y_n \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \ldots + a_{2n}y_n \\ &\vdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \ldots + a_{nn}y_n \end{aligned}$$

In **normal form**, this system can be written as:

$$\boxed{\vec{y}'(t) = A\,\vec{y}(t)} \qquad (1)$$

where

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{pmatrix} \quad , \quad A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Suppose that $\det A \neq 0$.

- **Consider the case when *n* = 1**. In such case our system reduces to a single scalar equation, so we have:

  $$y' = a\,y$$

  This is a first order linear homogeneous equation with constant coefficients ($y' - ay = 0$), so we seek the solution in the form $y = e^{rt}$, and after we substitute this into the equation, we would obtain $r = a$, so the general solution would be of the form

  $$\boxed{y = ce^{at}}$$

- **Consider the case when *n* > 1.** In the same manner as in the one-dimensional case, we can seek equilibrium solutions for the higher-dimensional case $\vec{y}' = A\,\vec{y}$**.** Since we have a homogeneous system with constant coefficients, we will seek the solution in the form:

  $$\boxed{\vec{y} = \vec{v}\,e^{\lambda t}} \quad \Rightarrow \quad \vec{y}' = \vec{v}\,\lambda e^{\lambda t}$$

  Substituting this into the above system (1), <u>we obtain the eigenvalue problem of the matrix **A**</u>:

  $$\left.\begin{aligned} \vec{y} &= \vec{v}\,e^{\lambda t} \\ \vec{y}' &= \vec{v}\,\lambda e^{\lambda t} \end{aligned}\right\} \;\Rightarrow\; A\,\vec{y} = \vec{y}' \;\Leftrightarrow\; A\,\vec{v}\,e^{\lambda t} = \vec{v}\,\lambda e^{\lambda t} \;\Leftrightarrow\; \underline{\underline{A\,\vec{v} = \lambda\,\vec{v}}}$$

Definition: Eigenvalues and Eigenvectors

Let $A=[a_{ij}]$ be a square matrix of size $n\times n$, with constant coefficients. Consider the equation:

$$A\vec{v}=\lambda\vec{v}$$

Then

- The (real or complex) numbers λ represent the eigenvalues of A.
- The associated nontrivial vectors $\vec{v}$ represent the eigenvectors of A.

This is a homogeneous system of the form $Mx=0$, which has nontrivial solutions provided that $M=A-\lambda I$ is a singular matrix. Hence, we assume that $\det M=\det(A-\lambda I)=0$.

How do we solve an eigenvalue problem?

$$A\vec{v}=\lambda\vec{v} \quad\Leftrightarrow\quad A\vec{v}-\lambda\vec{v}=\vec{0}$$

$$\Leftrightarrow\quad (A-\lambda I)\vec{v}=\vec{0}$$

$$\Rightarrow\quad \det(A-\lambda I)=0$$

A is an $n\times n$ matrix, so the equation $\det(A-\lambda I)=0$ reduces to finding the roots of an n^{th} degree polynomial. This equation is called the **characteristic equation** of A. The polynomial $p(r)=\det(A-\lambda I)$ represents the **characteristic polynomial** of A.

How to apply the solutions of the eigenvalue problem to a system of DEs?

Let A be a $n\times n$ constant matrix with n linearly independent eigenvectors $\vec{v}_1$, $\vec{v}_2,\ldots,\vec{v}_n$ and associated eigenvalues λ_1, $\lambda_2,\ldots,\lambda_n$. Then, for a homogeneous system $\vec{y}'(t)=A\,\vec{y}(t)$, the fundamental solution set of on $(-\infty,\infty)$ is the set

$$\left\{e^{\lambda_1 t}\,\vec{v}_1, e^{\lambda_2 t}\,\vec{v}_2,\ldots, e^{\lambda_n t}\,\vec{v}_n\right\}$$

so the general solution is of the form

$$\boxed{\vec{y}(t)=c_1\cdot e^{\lambda_1 t}\,\vec{v}_1 + c_2\cdot e^{\lambda_2 t}\,\vec{v}_2+\ldots+c_n\cdot e^{\lambda_n t}\,\vec{v}_n}$$

where c_1, c_2, …, c_n are arbitrary constants.

Ex.1 Consider the following system in a matrix form:

$$\vec{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \vec{y}\ , \quad \vec{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

a) Find the eigenvalues and the eigenvectors of the matrix $A = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix}$.

$$A\vec{v} = \lambda\vec{v} \quad \Rightarrow \quad A\vec{v} - \lambda I \vec{v} = \vec{0}$$

$$\Leftrightarrow \quad (A - \lambda I)\vec{v} = \vec{0}$$

$$\Leftrightarrow \quad \left(\begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \vec{v} = \vec{0}$$

$$\Leftrightarrow \quad \underbrace{\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix}}_{M} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

A homogeneous system has nontrivial solutions if the determinant of the system is equal to zero:

$$\det M = 0 \quad \Rightarrow \quad \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} = 0$$

$$\begin{vmatrix} -(\lambda-2) & 4 \\ 4 & -(\lambda-2) \end{vmatrix} = (\lambda-2)^2 - 16 = 0$$

$$\lambda^2 - 4\lambda + 4 - 16 = 0$$

$$\lambda^2 - 4\lambda - 12 = 0$$

$$(\lambda - 6)(\lambda + 2) = 0 \quad \nearrow \lambda_1 = 6, \quad \searrow \lambda_2 = -2$$

Next, for each of the two eigenvalues we find a corresponding eigenvector.

$\lambda = 6$

$$\begin{bmatrix} 2-6 & 4 \\ 4 & 2-6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-4x_1 + 4x_2 = 0$$
$$4x_1 - 4x_2 = 0$$

$$x_1 = x_2$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 2-(-2) & 4 \\ 4 & 2-(-2) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$4x_1 + 4x_2 = 0$$
$$4x_1 + 4x_2 = 0$$

$$x_1 = -x_2$$

$$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b) Using the solution to the eigenvalue problem of the given matrix that you found in the previous part, determine the solution to the following IVP:

$$\vec{y}\,' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \vec{y} \quad , \quad \vec{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

The two solutions from the previous part are:

$$\vec{y}_1 = \vec{v}_1 \cdot e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$
$$\vec{y}_2 = \vec{v}_2 \cdot e^{\lambda_2 t} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

Thus the general solution is

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

We determine the constants from the initial condition for the vector $\vec{y}$.

$$\vec{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}$$

$$c_1 - c_2 = 5$$
$$c_1 + c_2 = -1$$
$$2c_1 = 4 \Rightarrow c_1 = 2$$
$$c_2 = -1 - c_1 \Rightarrow c_2 = -3$$

Thus,

$$\vec{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t} = \begin{bmatrix} 2e^{6t} + 3e^{-2t} \\ 2e^{6t} - 3e^{-2t} \end{bmatrix}$$

The two 1-dimensional solutions (corresponding to the components of the vector solution $\vec{y}$) are:

$$y_1(t) = 2e^{6t} + 3e^{-2t}$$
$$y_2(t) = 2e^{6t} - 3e^{-2t}$$

Ex. 2 Consider the following 2-dimensional initial-value problem:

$$\vec{y}\,' = \begin{pmatrix} 5 & -1 \\ 3 & 1 \end{pmatrix} \vec{y} \quad , \quad \vec{y}\,'(0) = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

a) Write the system in expanded form so that you have two differential equations of first order, with two initial conditions.

$$(1) \quad y_1' = 5y_1 - y_2 \; , \quad y_1(0) = 2$$
$$(2) \quad y_2' = 3y_1 + y_2 \; , \quad y_2(0) = -1$$

b) Combine the equations to obtain a second order IVP, in terms of one of the functions. Solve the IVP and then use the result to determine the solution for the other function.

From (1):

$$y_2 = 5y_1 - y_1' \;\Rightarrow\; y_2' = 5y_1' - y_1''$$

Substitute into (2):

$$y_2' = 3y_1 + y_2 \;\Leftrightarrow\; (5y_1' - y_1'') = 3y_1 + (5y_1 - y_1')$$
$$\Leftrightarrow\; -y_1'' + 6y_1' - 8y_1 = 0$$
$$\Leftrightarrow\; y_1'' - 6y_1' + 8y_1 = 0 \; , \quad y_1(0) = 2, \; y_1'(0) = 11$$

Initial conditions are obtained from the system: $y_1(0) = 2$ and $y_1'(0) = 5y_1(0) - y_2(0) = 5(2) - (-1) = 11$

Solve the second order IVP:

$$y_1 = e^{rt} \Rightarrow y_1' = re^{rt}, \quad y_1'' = r^2e^{rt} \quad \Big\} \quad (r^2 - 6r + 8)e^{rt} = 0$$
$$(r-4)(r-2) = 0 \quad \nearrow r_1 = 4 \quad y_{11} = e^{4t}; \quad \searrow r_2 = 2 \quad y_{12} = e^{2t}$$

$$y_1 = c_1e^{4t} + c_2e^{2t} \qquad y_1(0) = 2 \Rightarrow c_1 + c_2 = 2 \quad /(-2)$$
$$y_1' = 4c_1e^{4t} + 2c_2e^{2t} \qquad y_1'(0) = 11 \Rightarrow 4c_1 + 2c_2 = 11$$
$$2c_1 = 7 \Rightarrow c_1 = \frac{7}{2}$$
$$c_2 = 2 - c_1 \Rightarrow c_2 = 2 - \frac{7}{2} = -\frac{3}{2}$$

Thus,

$$y_1(t) = \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t}$$

Back-solve for the second function:

$$y_2 = 5y_1 - y_1' = 5\left(\frac{7}{2}e^{4t} - \frac{3}{2}e^{2t}\right) - \left(14e^{4t} - 3e^{2t}\right)$$
$$= \left(\frac{35}{2} - 14\right)e^{4t} + \left(-\frac{15}{2} + 3\right)e^{2t}$$
$$= \frac{7}{2}e^{4t} - \frac{9}{2}e^{2t}$$

c) Find the eigenvalues and the eigenvectors of the matrix $A=\begin{pmatrix}5 & -1\\ 3 & 1\end{pmatrix}$.

- Matrix eigenvalue problem:

$$A\vec{v}=\lambda\cdot\vec{v}$$

$$\Leftrightarrow (A-\lambda I)\vec{v}=0 \quad \Rightarrow \quad \boxed{\det(A-\lambda I)=0}$$

$$\Rightarrow \begin{vmatrix}5-\lambda & -1\\ 3 & 1-\lambda\end{vmatrix} = \begin{vmatrix}-(\lambda-5) & -1\\ 3 & -(\lambda-1)\end{vmatrix} = (\lambda-5)(\lambda-1)+3=0$$

Hence,

$$\lambda^2-6\lambda+8=0$$

$$(\lambda-4)(\lambda-2)=0 \quad \begin{array}{l}\nearrow \boxed{\lambda_1=4}\\ \searrow \boxed{\lambda_2=2}\end{array}$$

$\underline{\lambda_1=4}$, $\vec{v}_1=?$

$$(A-\lambda I)\cdot\vec{v}=0$$

$$\begin{pmatrix}5-4 & -1\\ 3 & 1-4\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}1 & -1\\ 3 & -3\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$x_1-x_2=0$$

$$x_1=x_2 \quad \Rightarrow \quad \boxed{\vec{v}_1=\begin{pmatrix}1\\ 1\end{pmatrix}}$$

$\underline{\lambda_2=2}$, $\vec{v}_2=?$

$$(A-\lambda I)\vec{v}=0$$

$$\begin{pmatrix}5-2 & -1\\ 3 & 1-2\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$\begin{pmatrix}3 & -1\\ 3 & -1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$$

$$3x_1-x_2=0$$

$$x_2=3x_1 \quad \Rightarrow \quad \boxed{\vec{v}_2=\begin{pmatrix}1\\ 3\end{pmatrix}}$$

The two eigenpairs are:

$\lambda_1=4$, $\vec{v}_1=\begin{pmatrix}1\\ 1\end{pmatrix}$

$\lambda_2=2$, $\vec{v}_2=\begin{pmatrix}1\\ 3\end{pmatrix}$

d) Use the result from the previous part to solve the given IVP in matrix form.

$$\vec{y}\,'=\begin{pmatrix}5 & -1\\ 3 & 1\end{pmatrix}\vec{y}, \quad \vec{y}(0)=\begin{pmatrix}2\\ -1\end{pmatrix}$$

Recall, to solve

$$\boxed{\vec{y}\,'=A\vec{y}} \qquad (\square)$$

we seek the solution as:

$$\vec{y}=\vec{v}\cdot e^{\lambda t} \quad \Rightarrow \quad \vec{y}\,'=\vec{v}\cdot\lambda e^{\lambda t}$$

Substituting this into ($\square$), we obtain the eigenvalue problem of the matrix A:

$$\vec{y}\,'=A\vec{y}$$

$$\Leftrightarrow \vec{v}\cdot\lambda e^{\lambda t}=A(\vec{v}\cdot e^{\lambda t})$$

$$\Leftrightarrow (\lambda\cdot\vec{v})\cdot e^{\lambda t}=(A\vec{v})\,e^{\lambda t}$$

$$\Leftrightarrow \underline{A\vec{v}=\lambda\cdot\vec{v}}$$

Thus, the value of the parameter in (□) is an eigenvalue λ of the matrix A while the vector $\vec{v}$ corresponds to an eigenvector of the matrix A. From the previous part we have

$$\lambda_1 = 4, \quad \vec{v}_1 = \begin{pmatrix}1\\1\end{pmatrix}$$
$$\lambda_2 = 2, \quad \vec{v}_2 = \begin{pmatrix}1\\3\end{pmatrix}$$

Hence, the general solution is

$$\vec{y}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} = \boxed{c_1 \begin{pmatrix}1\\1\end{pmatrix} e^{4t} + c_2 \begin{pmatrix}1\\3\end{pmatrix} e^{2t}}$$

We also need to determine the constants:

$$\vec{y}(0) = \begin{pmatrix}2\\-1\end{pmatrix} \Rightarrow \quad c_1 \begin{pmatrix}1\\1\end{pmatrix} + c_2 \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix}$$

$$c_1 + c_2 = 2 \quad /\cdot(-1)$$
$$c_1 + 3c_2 = -1$$
$$2c_2 = -3 \Rightarrow \quad c_2 = -\frac{3}{2}$$
$$c_1 = 2 - c_2 \Rightarrow \quad c_1 = 2 + \frac{3}{2} \qquad c_1 = \frac{7}{2}$$

Thus, the vector solution is:

$$\boxed{\vec{y}(t) = \frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix} e^{4t} - \frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix} e^{2t}}$$

The single solutions are:

$$\vec{y}(t) = \begin{pmatrix} \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t} \\ \frac{7}{2}e^{4t} - \frac{9}{2}e^{2t} \end{pmatrix} \Rightarrow \quad \begin{aligned} y_1(t) &= \frac{7}{2}e^{4t} - \frac{3}{2}e^{2t} \\ y_2(t) &= \frac{7}{2}e^{4t} - \frac{9}{2}e^{2t} \end{aligned}$$

which matches the earlier obtained solutions.

e) Verify that the vector you obtained is a solution to the given IVP.
First, we need to verify that the solution satisfies the initial condition:

$$\vec{y}(0) = \frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix} - \frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}7/2\\7/2\end{pmatrix} - \begin{pmatrix}3/2\\9/2\end{pmatrix} = \begin{pmatrix}4/2\\-2/2\end{pmatrix} = \begin{pmatrix}2\\-1\end{pmatrix} \checkmark$$

Next, we need to verify that the solution satisfies the matrix equation $\vec{y}' = A\vec{y}$.

LHS:
$$\vec{y}' = \frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix}\cdot 4e^{4t} - \frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix}\cdot 2e^{2t}$$
$$= 14\begin{pmatrix}1\\1\end{pmatrix}e^{4t} - 3\begin{pmatrix}1\\3\end{pmatrix}e^{2t}$$
$$= \begin{pmatrix}14\\14\end{pmatrix}e^{4t} - \begin{pmatrix}3\\9\end{pmatrix}e^{2t}$$

RHS:
$$A\vec{y} = \begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}\vec{y}$$
$$= \begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}\left(\frac{7}{2}\begin{pmatrix}1\\1\end{pmatrix}e^{4t} - \frac{3}{2}\begin{pmatrix}1\\3\end{pmatrix}e^{2t}\right)$$
$$= \frac{7}{2}\cdot\begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}\begin{pmatrix}1\\1\end{pmatrix}e^{4t} - \frac{3}{2}\cdot\begin{pmatrix}5 & -1\\3 & 1\end{pmatrix}\begin{pmatrix}1\\3\end{pmatrix}e^{2t}$$
$$= \frac{7}{2}\cdot\begin{pmatrix}4\\4\end{pmatrix}e^{4t} - \frac{3}{2}\begin{pmatrix}2\\6\end{pmatrix}e^{2t}$$
$$= \begin{pmatrix}14\\14\end{pmatrix}e^{4t} - \begin{pmatrix}3\\9\end{pmatrix}e^{2t}$$

Additional Example

(Ex.) Find the general solution of:

$$\vec{x}'_{(t)} = A \cdot \vec{x}(t), \quad \text{for } A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}.$$

Solution:

Intro: $\vec{x}' = A\vec{x} \iff \vec{x}' - A\vec{x} = 0$ } homogeneous system with const. coefficients.

$\Rightarrow$ $\boxed{\vec{x} = \vec{v} \cdot e^{rt}}$ – we seek the solution in this form ($\vec{v}$=const)

$\Rightarrow$ $\vec{x}' = \vec{v} \cdot r e^{rt}$

$$\vec{v} \cdot r e^{rt} = A \cdot (\vec{v} e^{rt})$$

$\iff$ $r\vec{v} = A\vec{v}$

$\iff$ $\boxed{A\vec{v} = r \cdot \vec{v}}$ } The solution vectors are the eigenvectors of A

Eigenproblem of A

$A = \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix}$ $\qquad A\vec{v} = r\vec{v}$

$\iff A\vec{v} - r\vec{v} = \vec{0}$

$\iff (A - rI) \cdot \vec{v} = \vec{0}$ } This homogeneous system has nontrivial solutions iff $\det(A-rI)=0$

$\Rightarrow \det(A - rI) = 0$

$$\Rightarrow \begin{vmatrix} 2-r & -3 \\ 1 & -2-r \end{vmatrix} = 0$$

$$\iff \begin{vmatrix} -(r-2) & -3 \\ 1 & -(r+2) \end{vmatrix} = 0$$

$\iff (r-2)(r+2) + 3 = 0$

$\iff r^2 - 4 + 3 = 0$

$\iff r^2 - 1 = 0 \Rightarrow r_1 = 1,\ r_2 = -1$

Once we have the eigenvalues, we substitute them into the eigenvalue equation:

$$A\vec{v} = r\vec{v} \iff \begin{bmatrix} 2 & -3 \\ 1 & -2 \end{bmatrix} \vec{v} = r \cdot \vec{v}$$

$$\iff \begin{bmatrix} 2-r & -3 \\ 1 & -2-r \end{bmatrix} \cdot \vec{v} = \vec{0}$$

$r_1 = 1$

$$\begin{bmatrix} 1 & -3 \\ 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad x_1 - 3x_2 = 0$

$\Rightarrow \quad x_1 = 3x_2$

Let $x_2 = s$, then $x_1 = 3s$, so

$$\vec{v}_1 = \begin{bmatrix} 3s \\ s \end{bmatrix} = s \cdot \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

setting $s = 1$, we obtain:

$$\boxed{\vec{v}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}}$$

$r_2 = -1$

$$\begin{bmatrix} 3 & -3 \\ 1 & -1 \end{bmatrix} \vec{v} = \vec{0}$$

$\Rightarrow \quad x_1 - x_2 = 0$

$\Rightarrow \quad x_1 = x_2$

let $x_2 = s$, then $x_1 = s$ so

$$\vec{v}_2 = \begin{bmatrix} s \\ s \end{bmatrix} = s \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

setting $s = 1$, we obtain

$$\boxed{\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$$\vec{x} = \vec{v} \cdot e^{rt} \Rightarrow \vec{x}_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \cdot e^{t} \text{ and } \vec{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot e^{-t}$$

The general solution is the linear combination:

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 \Rightarrow \boxed{\vec{x}(t) = c_1 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}}$$

Ex. Solve the given IVP:

$$\vec{X}'(t) = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{X}(t), \quad \vec{X}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Note: We seek solutions of the form $\vec{X} = \vec{v} \cdot e^{rt}$, where $(r, \vec{v})$ represents an eigenpair of the matrix A.

• Solving the eigenproblem of A:

$$A\vec{v} = r \cdot \vec{v} \iff (A - Ir)\vec{v} = \vec{0}$$

$$\Rightarrow \det(A - Ir) = 0$$

$$\Rightarrow \begin{vmatrix} 1-r & 3 \\ 3 & 1-r \end{vmatrix} = 0$$

$$\iff p(r) = (1-r)^2 - 9 = 0$$

$$\iff 1 - 2r + r^2 - 9 = 0$$

$$\iff r^2 - 2r - 8 = 0$$

$$\iff (r-4)(r+2) = 0 \quad \begin{cases} \boxed{r_1 = 4} \\ \boxed{r_2 = -2} \end{cases}$$

$\vec{v}_1 = ?$ $\quad r_1 = 4$

$$(A - Ir) \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff 3x_1 - 3x_2 = 0$$

$$\iff x_1 - x_2 = 0$$

$$\Rightarrow x_1 = x_2 = s$$

$$\Rightarrow \vec{v}_1 = s \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ or } \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

$\vec{v}_2 = ?$ $\quad r_2 = -2$

$$(A - Ir) \cdot \vec{v} = \vec{0}$$

$$\begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\iff 3x_1 + 3x_2 = 0$$

$$\iff x_1 = -x_2 = -s$$

$$\vec{v}_2 = \begin{bmatrix} -s \\ s \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \end{bmatrix} \Rightarrow \boxed{\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$$\vec{X} = c_1 \cdot \vec{v}_1 \cdot e^{r_1 t} + c_2 \vec{v}_2 \cdot e^{r_2 t} \Rightarrow \boxed{\vec{X}(t) = c_1 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

$c_1, c_2 = ?$

$$\vec{x}(0) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} \Rightarrow c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{bmatrix} c_1 - c_2 \\ c_1 + c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\Leftrightarrow \begin{aligned} c_1 - c_2 &= 3 \\ c_1 + c_2 &= 1 \end{aligned}$$

$$2c_1 = 4 \Rightarrow c_1 = 2$$

$$c_2 = 1 - c_1 \Rightarrow c_2 = -1$$

$$\boxed{\vec{x}(t) = 2 \cdot e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} - e^{-2t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

or

$$\vec{x}(t) = \begin{bmatrix} 2e^{4t} + e^{-2t} \\ 2e^{4t} - e^{-2t} \end{bmatrix}$$

References: Trench, William F. (2013). *Elementary Differential Equations.* Retrieved from Digital Commons: http://digitalcommons.trinity.edu/mono/8