10.3 Basic Theory of Homogenous Linear Systems

GENERAL SOLUTION FOR HOMOGENEOUS CASE

- We consider the homogeneous linear system

$$\vec{y}'(t) = A(t)\vec{y}(t) \quad (1)$$

where $A(t)$ is a continuous matrix function of size $n \times n$ on an interval (a,b).

- A **trivial** solution of the system (1) is the zero vector $\vec{y}(t) \equiv \vec{0}$. Any other solution is **nontrivial**.
- If $\vec{y}_1$, $\vec{y}_2$, ..., $\vec{y}_n$ are linearly independent solutions of the system (1) on (a,b), then $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ is a **fundamental set of solutions** and this set defines the general solution as:

$$\boxed{\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n}$$

where c_1, c_2, ..., c_n are arbitrary constants.

Theorem: Consider the homogeneous linear system

$$\vec{y}'(t) = A(t)\vec{y}(t) \quad (2)$$

where $A(t)$ is a continuous matrix function of size $n \times n$ on (a,b). The following are equivalent:

- The general solution to the homogeneous system (1) on (a,b) is $\vec{y}(t) = c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n$
- $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ is a fundamental set of solutions of the homogenous system (1)
- $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ is linearly independent on (a,b)
- The Wronskian of $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ is nonzero at some point in (a,b).

The Wronskian of $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ is nonzero at all points in (a,b).

The Wronskian of a set of n vector functions $\vec{y}_1$, $\vec{y}_2$, ..., $\vec{y}_n$ is the real-valued function

$$W(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n) = \begin{vmatrix} | & | & & | \\ \vec{y}_1 & \vec{y}_2 & \cdots & \vec{y}_n \\ | & | & & | \end{vmatrix} = \begin{vmatrix} y_{11} & y_{12} & \cdots & y_{1n} \\ y_{21} & y_{22} & \cdots & y_{2n} \\ \vdots & \vdots & & \vdots \\ y_{n1} & y_{n2} & \cdots & y_{nn} \end{vmatrix}$$

GENERAL SOLUTION FOR NONHOMOGENEOUS CASE

Let $\vec{y}_p(t)$ be a particular solution of the nonhomogeneous system

$$\boxed{\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t)} \quad (3)$$

on some interval I and let $\{\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n\}$ be a fundamental solution set on I for the corresponding homogeneous system $\vec{y}'(t) = A(t)\vec{y}(t)$. Then every solution of (3) can be written as

$$\boxed{\vec{y}(t) = \vec{y}_p + c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n}$$

where c_1, c_2, ..., c_n are constants.

Example:

1. The vector functions

$$\vec{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \vec{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system:

$$\vec{y}\,' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \vec{y}$$

on $(-\infty, \infty)$.

a) Verify that the two vector functions are linearly independent. *Hint*: Compute the Wronskian of $\{\vec{y}_1, \vec{y}_2\}$

$$W(t) = \begin{vmatrix} | & | \\ \vec{y}_1 & \vec{y}_2 \\ | & | \end{vmatrix} = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = -e^{t} + 2e^{t} = e^{t} \neq 0$$

b) Write the general solution of the above homogeneous system.

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

c) Solve the initial value problem $\vec{y}\,' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \vec{y}$, $\vec{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$

$$\vec{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix} \quad \Rightarrow \quad c_1 \vec{y}_1(0) + c_2 \vec{y}_2(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

$$-c_1 - c_2 = 4$$

$$2c_1 + c_2 = -5$$

$$c_1 = -1$$

$$c_2 = -5 - 2c_1 \quad \Rightarrow \quad c_2 = -3$$

Thus,

$$\vec{y} = c_1 \vec{y_1} + c_2 \vec{y_2}$$

$$\vec{y} = - \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} \\ -2e^{2t} \end{bmatrix} + \begin{bmatrix} 3e^{-t} \\ -3e^{-t} \end{bmatrix}$$

$$\vec{y} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

Abel's Formula: Let $A(t)$ be a continuous matrix function of size $n \times n$ on an interval (a,b) and let $\{\vec{y}_1, \vec{y}_2, ..., \vec{y}_n\}$ be solutions of the homogeneous system (1) on (a,b). Then, for some t_0 in (a,b), the Wronskian of these vector functions can be expressed as:

$$W(t) = W(t_0)\exp\left(\int_{t_0}^{t}[a_{11}(s) + a_{22}(s) + ... + a_{nn}(s)]ds\right) = W(t_0)e^{\int_{t_0}^{t} tr(A(s))ds}$$

Thus, the Wronskian either has no zeros in (a,b) or it is zero at all points of (a,b).

2. Verify Abel's formula for the homogeneous system from the previous exercise.

First, determine the trace of the given matrix:

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \Rightarrow tr(A) = -4 + 5 = 1$$

Then,

$$W(t) = W(t_0) \cdot e^{\int_{t_0}^{t} trA\,ds} = W(t_0) \cdot e^{\int_{t_0}^{t} ds}$$

$$W(t) = W(t_0) \cdot e^{t-t_0} = e^{t_0} \cdot e^{t-t_0} = e^{t}$$

which corresponds to the result obtained in part a) from previous exercise.

Additional Example

1. Show that the given vector functions: are linearly independent on $(-\infty, \infty)$.

$$\vec{y}_1(t) = \begin{bmatrix} e^{t} \\ 0 \\ e^{-t} \end{bmatrix}, \quad \vec{y}_2(t) = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \vec{y}_3(t) = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

Answer: it suffices to show that the Wronskian determinant is non-zero:

$$W = \begin{vmatrix} | & | & | \\ \vec{y}_1 & \vec{y}_2 & \vec{y}_3 \\ | & | & | \end{vmatrix} = \begin{vmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix}$$

$$W = e^{t}(-1)^{1+1} \cdot \begin{vmatrix} e^{3t} & e^{3t} \\ 1 & 0 \end{vmatrix} + 0 + e^{2t} \cdot (-1)^{1+3} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix}$$

$$W = e^{t}(-e^{3t}) + e^{2t}(-e^{2t})$$

$$W = -e^{4t} - e^{4t}$$

$$W = -2e^{4t} \neq 0$$

How do we know that this ensures linear independence? By its definition, linear independence means

$$\vec{y} = c_1 \vec{y}_1 + c_2 \vec{y}_2 + c_3 \vec{y}_3 = \vec{0} \quad \Leftrightarrow \quad c_1 = c_2 = c_3 = 0$$

Then,

$$c_1 \begin{bmatrix} e^{t} \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} e^{t} & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A homogeneous system with a nonzero determinant only has a trivial solution, so we must have $c_1 = c_2 = c_3 = 0$.

References: Trench, William F. (2013). *Elementary Differential Equations.* Retrieved from Digital Commons: http://digitalcommons.trinity.edu/mono/8