Sec. 10.2 Linear Systems of Differential Equations

NONHOMOGENEOUS SYSTEM

- A **linear system** of simultaneous ordinary differential equations has the general form

$$\begin{aligned}
y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n + f_1(t) \\
y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n + f_2(t) \\
&\vdots \\
y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n + f_n(t)
\end{aligned}$$

In matrix form (*normal form*), this system can be rewritten as:

$$\begin{bmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{bmatrix} = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} + \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$$

of equations reduces to a single matrix equation:

$$\boxed{\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t) \quad , \quad \vec{y}(t_0) = \vec{k}}$$

Theorem (uniqueness): If the coefficient matrix $A(t)$ and the forcing vector function $\vec{f}(t)$ are continuous[1] on an open interval (a,b) which contains the initial point t_0 and $\vec{k}$ is an arbitrary vector of n constants, then the initial value problem

$$\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t) \quad , \quad \vec{y}(t_0) = \vec{k}$$

has a unique solution $\vec{y}(t)$ on the given interval (a,b).

HOMOGENEOUS SYSTEM

- If each function $f_k(t)$ equals zero on the interval $I = (a,b)$, the above system is **homogeneous**,

$$\begin{aligned}
y_1' &= a_{11}(t)y_1 + a_{12}(t)y_2 + \dots + a_{1n}(t)y_n \\
y_2' &= a_{21}(t)y_1 + a_{22}(t)y_2 + \dots + a_{2n}(t)y_n \\
&\vdots \\
y_n' &= a_{n1}(t)y_1 + a_{n2}(t)y_2 + \dots + a_{nn}(t)y_n
\end{aligned}$$

In normal form, we have

$$\boxed{\vec{y}'(t) = A(t)\vec{y}(t) \quad , \quad \vec{y}(t_0) = \vec{k}}$$

[1] Continuity of a matrix or a column-vector means that each entry is a continuous function.

1. Consider the system:

$$y_1' = y_1 + 2y_2 + 2e^{4t}$$
$$y_2' = 2y_1 + y_2 + e^{4t}$$

a) Write the system in matrix form and using the previous Theorem conclude that every IVP for this system has a unique solution on $(-\infty, \infty)$.

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$$

$$\vec{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \vec{y}(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}$$

Observe that the coefficient matrix and the forcing term are continuous on $(-\infty, \infty)$, so a unique solution to an IVP exists on $(-\infty, \infty)$.

b) Verify that $\vec{y} = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ is a solution for all constant c_1 and c_2.

We want to verify that the equation $\vec{y}'(t) = A(t)\vec{y}(t) + \vec{f}(t)$ holds. We can separately express each the LHS and RHS of the equation.

$$\vec{y}' = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} \cdot 4e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 3e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} (-e^{-t})$$
$$= \frac{1}{5}\begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

Also,

$$A\vec{y} + \vec{f} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \left(\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \right) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$$
$$= \frac{1}{5}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$$
$$= \frac{1}{5}\begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}$$
$$= \frac{1}{5}\begin{bmatrix} 22+10 \\ 23+5 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$
$$= \frac{1}{5}\begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t}$$

c) Find the solution to the IVP

$$\vec{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \vec{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \quad , \quad \vec{y}(0) = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}$$

Now we need to solve for the constants, using the initial condition.

$$\vec{y}(0) = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4(0)} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3(0)} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-(0)} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}$$

$$\frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} + \begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 \\ 22 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5 \\ 15 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ c_1 - c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$c_1 + c_2 = -1$$
$$c_1 - c_2 = 3$$

$$2c_1 = 2 \Rightarrow \boxed{c_1 = 1}$$

$$c_2 = -1 - c_1 \Rightarrow \boxed{c_2 = -2}$$

Thus,

$$\boxed{\vec{y}(t) = \frac{1}{5}\begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + 1 \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}}$$

References: Trench, William F. (2013). *Elementary Differential Equations.* Retrieved from Digital Commons: http://digitalcommons.trinity.edu/mono/8