Linear Systems of Differential Equations

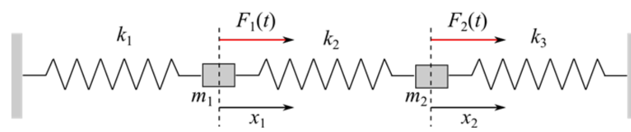
Goal: using methods of matrix algebra, solve systems of linear differential equations

Sec. 10.1 Introduction to Systems of DEs

In many physical problems separate elements are linked together (e.g. a mechanical system of connected springs or a network of electrical circuits). In such cases our mathematical model is represented by a system of two or more differential equations.

Ex. A mechanical system of three springs with constants k_1, k_2, k_3 and two masses m_1, m_2 is described by the following system of differential equations:

$$m_1 y_1'' = -k_1 y_1 + k_2 (y_2 - y_1) + F_1$$
$$m_2 y_2'' = -k_3 y_2 - k_2 (y_2 - y_1) + F_2$$



System of first order ODEs

A system of simultaneous 1st order ordinary differential equations if of the general form:

$$\begin{aligned} y_1' &= g_1(t, y_1, y_2, \ldots y_n) \\ y_2' &= g_2(t, y_1, y_2, \ldots y_n) \\ &\vdots \\ y_n' &= g_n(t, y_1, y_2, \ldots y_n) \end{aligned}$$

where each y_k $(k = 1, 2, \ldots, n)$ is a function of t. The system is LINEAR if each g_k is a linear function of $(y_1, y_2, \ldots y_n)$ and otherwise the system is NONLINEAR.

Transforming a DE into a system of 1st order DEs

- Every differential equation of 2nd or higher order can be transformed into a system of first order differential equations.

- In general, an arbitrary nth order DE of the form

$$y^{(n)} = F\left(t, y, y', y'', \ldots, y^{(n-1)}\right)$$

can be transformed into a system of n first order equations by setting

$$y_1 = y,\ y_2 = y',\ y_3 = y'', \ldots, y_n = y^{(n-1)}$$

in which case the system becomes:

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ &\vdots \\ y_{n-1}' &= y_n \\ y_n' &= F(t, y_1, y_2, \ldots y_n) \end{aligned}\right\} \Leftrightarrow \boxed{\vec{y}' = A\vec{y}} - \textit{normal form}$$

Examples

1. (Based on Ex. 5-7, Sec. 10.1) Rewrite the given DE as a system of 1st order DEs. Express the system in matrix form.

a) $y''(t)+5y'(t)-7y(t)=0$

Answer:

Rewrite the DE by expressing the highest derivative y'':

$$\boxed{y'' = \underset{\substack{\uparrow\\ y_1}}{7y} - \underset{\substack{\uparrow\\ y_2}}{5y'}}$$

Introduce y_1 and y_2 such that:

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\}$$

Consider the derivatives of y_1 and y_2

That way we obtain a system:

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= 7y_1 - 5y_2 \end{aligned}} \quad \Leftrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 7 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

b) $y''(t)+8y'(t)-5y(t)=2\cos(3t)$

Answer:

Express y''

$$y'' = \underset{\substack{\uparrow\\ y_1}}{5y} - \underset{\substack{\uparrow\\ y_2}}{8y'} + 2\cos 3t$$

Let:

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \end{aligned}\right\} \quad \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= 5y_1 - 8y_2 + 2\cos(3t) \end{aligned}}$$

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ 5 & -8 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\cos(3t) \end{bmatrix}$$

2. Express the DE as a system of first order equations and in matrix notation (normal form).

$$y^{(4)} + 4y'''(t) + 6y''(t) + 11y'(t) + y(t) = 7\sin t + e^{3t}$$

Answer:

Express the highest order derivative:

$$y^{(4)} = -\underset{\substack{\uparrow \\ y_1}}{y} - 11\underset{\substack{\uparrow \\ y_2}}{y'} - 6\underset{\substack{\uparrow \\ y_3}}{y''} - 4\underset{\substack{\uparrow \\ y_4}}{y'''} + 7\sin t + e^{3t}$$

Let:

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ y_4 &= y''' \end{aligned}\right\} \Rightarrow \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -y_1 - 11y_2 - 6y_3 - 4y_4 + 7\sin t + e^{3t} \end{aligned}$$

or, in normal (matrix) form,

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & -11 & -6 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 7\sin t + e^{t} \end{bmatrix}$$

- A **system** of two differential equations of 2^{nd} or higher order can be transformed into a system of first order differential equations.

3. Express the system in normal form.

$$x'' + 6x - 5y = 0$$
$$y'' - 3x + 7y = 0$$

Answer: First, rewrite the DEs so that the highest order derivatives are expressed:

$$x'' = -6x + 5y$$
$$y'' = \ 3x - 7y$$

Taking into account x, x', y, and y', we view our system as:

$$x'' = -6x + 0 \cdot x' + 5y + 0 \cdot y'$$
$$y'' = \ 3x + 0 \cdot x' - 7y + 0 \cdot y'$$

Introduce y_1, y_2, y_3, and y_4, such that:

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x' \\ y_3 &= y \\ y_4 &= y' \end{aligned}\right\}$$

That way we obtain a system:

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -6y_1 + 5y_3 \\ y_3' &= y_4 \\ y_4' &= 3y_1 - 7y_3 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -6 & 0 & 5 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -7 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

4. Express the system in normal form.

$$x'' - 3x' + 6t\,y + (\tan 2t)x = 0$$
$$y''' + 5y'' - t^3 x' + 2e^t x = 0$$

Since the 1^{st} DE features x'', the possible terms to consider are x and x'. Similarly, the 2^{nd} DE features y''', so we are taking into account y, y', and y''.

Answer: Again, express the highest derivatives first.

$$x'' = -\tan(2t) \cdot x + 3x' - 6t \cdot y$$
$$y''' = -2e^t \cdot \underset{\overset{\uparrow}{y_1}}{x} + t^3 \underset{\overset{\uparrow}{y_2}}{x'} + 0 \cdot \underset{\overset{\uparrow}{y_3}}{y} + 0\underset{\overset{\uparrow}{y_4}}{y'} - 5\underset{\overset{\uparrow}{y_5}}{y''}$$

Let:

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x' \\ y_3 &= y \\ y_4 &= y' \\ y_5 &= y'' \end{aligned}\right\} \quad \begin{aligned} y_1' &= y_2 \\ y_2' &= -\tan(2t) \cdot y_1 + 3y_2 - 6t \cdot y_3 \\ y_3' &= y_4 \\ y_4' &= y_5 \\ y_5' &= -2e^t y_1 + t^3 y_2 - 5y_5 \end{aligned}$$

or:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -\tan(2t) & 3 & -6t & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -2e^t & t^3 & 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

Ex. Write the following differential equation as a system of two first-order equations:

$$y''(t) + 4y'(t) - y(t) = 0$$

Then express the system in matrix form.

Answer: Rewrite the DE by expressing the highest derivative y'':

$$\boxed{y'' \;=\; \underset{\substack{\uparrow \\ y_1}}{y} \;-\; 4\underset{\substack{\uparrow \\ y_2}}{y'}}$$

Introduce y_1 and y_2 such that:

$$y_1 = y$$
$$y_2 = y'$$

Consider the derivatives of y_1 and y_2

That way we obtain a system:

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -y_1 - 4y_2 \end{aligned}} \quad \Leftrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -1 & -4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Ex. Express the DE for the undamped, unforced mass-spring oscillator

$$my'' + ky = 0$$

as an equivalent system of first-order equations.

Answer: Rewrite the DE by expressing the highest derivative y'':

Introduce y_1 and y_2 such that:

$$y_1 = y$$
$$y_2 = y'$$

Consider the derivatives of y_1 and y_2

That way we obtain a system:

$$\boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= -\tfrac{k}{m} y_1 \end{aligned}} \quad \Leftrightarrow \quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Ex. Rewrite the DE as a system of first order equations:

$$y'''(t) - 4y''(t) + 6y'(t) - 9y(t) = 0$$

and then express the system in matrix notation (normal form).

Answer: Rewrite the DE by expressing the highest derivative y'':

$$\boxed{y''' = \underset{\uparrow \atop y_1}{9y} - \underset{\uparrow \atop y_2}{6y'} + \underset{\uparrow \atop y_3}{4y''}}$$

Introduce y_1 and y_2 such that:

$$\left.\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \end{aligned}\right\}$$

Consider the derivatives of y_1, y_2, y_3

That way we obtain a system:

$$\left.\right\} \boxed{\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= 9y_1 - 6y_2 + 4y_3 \end{aligned}} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 9 & -6 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Ex. Consider a coupled mass-spring oscillator governed by the system:

$$\frac{d^2x}{dt^2} + 2x - y = 0$$

$$\frac{d^2y}{dt^2} + 2y - 3x = 0$$

Rewrite the system as a set of four first order DEs and express them in matrix notation (normal form).

Answer: First, rewrite the DEs so that the highest order derivatives are expressed:

$$\begin{aligned} x'' &= -2x + y \\ y'' &= \ \ 3x - 2y \end{aligned}$$

Taking into account x, x', y, and y', we think of our system as:

$$\begin{aligned} x'' &= -2x + 0 \cdot x' + y + 0 \cdot y' \\ y'' &= \ \ 3x + 0 \cdot x' - 2y + 0 \cdot y' \end{aligned}$$

Since the 1st DE features x'', the possible terms to consider are x and x'. Similarly, the 2nd DE features y'', so we are taking into account y and y'.

Introduce y_1, y_2, y_3, and y_4, such that:

$$\left.\begin{aligned} y_1 &= x \\ y_2 &= x' \\ y_3 &= y \\ y_4 &= y' \end{aligned}\right\}$$

That way we obtain a system:

$$\left.\begin{aligned} y_1' &= y_2 \\ y_2' &= -2y_1 + y_3 \\ y_3' &= y_4 \\ y_4' &= 3y_1 - 2y_3 \end{aligned}\right\} \Leftrightarrow \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 3 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$