§8.1 Arc Length

Arc Length formula

The length of a smooth* curve $y = f(x)$ on $[a,b]$	The length of a smooth curve $x = g(y)$, $c \le y \le d$
$\boxed{L = \int_a^b \sqrt{1+[f'(x)]^2}\,dx}$	$\boxed{L = \int_c^d \sqrt{1+[g'(y)]^2}\,dy}$

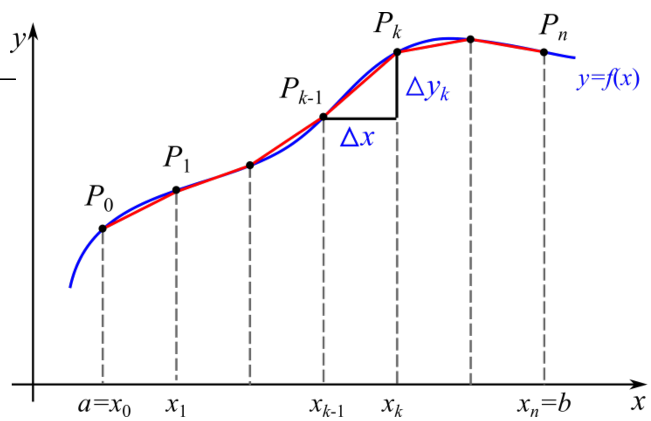
<u>Derivation</u>:

<u>Problem</u>: If f is a smooth function on $[a,b]$, what is the length of the curve $y = f(x)$ on $[a,b]$?

<u>Solution</u>:

- Divide the interval [*a*, *b*] into *n* subintervals of equal lengths Δx . Connect the points on the graph of the curve by line segments to obtain a polygonal path; the limit of this polygonal path approaches the arc length as $n \to \infty$, which corresponds to $\Delta x \to 0$.
- The length of the line segment on the k -th subinterval $[x_{k-1}, x_k]$ is

$$L_k = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(\Delta x)^2 + [f(x_k) - f(x_{k-1})]^2}$$



- By the Mean Value Theorem, there exists x_k^* in (x_{k-1}, x_k) such that

$$f'(x_k^*) = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$\Rightarrow \quad f'(x_k^*) = \frac{\Delta y_k}{\Delta x} \quad \text{or} \quad \Delta y_k = f'(x_k^*) \cdot \Delta x$$

- Thus,

$$L_k = \sqrt{(\Delta x)^2 + \left[f'(x_k^*)\right]^2 (\Delta x)^2} = \sqrt{1 + \left[f'(x_k^*)\right]^2}\;\Delta x$$

- The length of the curve is the limit of the polygonal path as $n \to \infty$ (or $\Delta x \to 0$)

$$L = \lim_{n\to\infty} \sum_{k=1}^{n} L_k = \lim_{n\to\infty} \sum_{k=1}^{n} \sqrt{1 + \left[f'(x_k^*)\right]^2}\;\Delta x$$

- In integral form:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx$$

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* **smooth** - a function f is smooth on some interval $[a,b]$ if it has a continuous derivative.

Arc Length Function:

The arc length of a curve from a particular starting point to *any* point on the curve:

$$\boxed{s(x)=\int_a^x \sqrt{1+\left[f'(t)\right]^2}\,dt}$$

$s(x)$ – the arc length function

$f(x)$ – smooth curve (continuous first derivative) on [*a*, *b*]

Differential of the Arc Length Function

By the Fundamental Theorem of Calculus* (first part), we can differentiate $s(x)$ to obtain

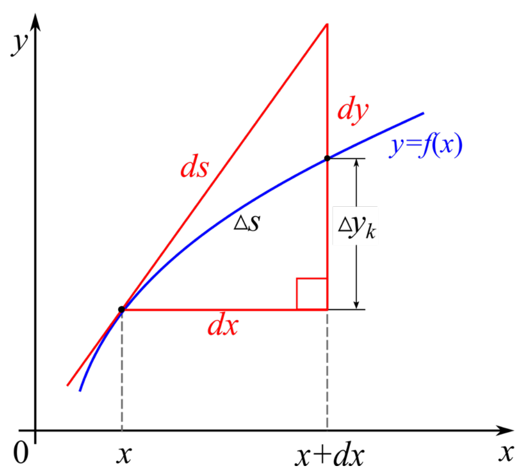
$$\boxed{\frac{ds}{dx}=\sqrt{1+[f'(x)]^2}}=\sqrt{1+\left[\frac{dy}{dx}\right]^2}$$

The equation shows important properties of ds/dx (the rate of change of *s* with respect to *x*):

- $\dfrac{ds}{dx}$ is always at least 1
- the value of $\dfrac{ds}{dx}$ is equal to 1 if the slope of the curve is equal to zero.

The differential of arc length is

$$ds=\sqrt{1+[f'(x)]^2}\,dx \quad\Rightarrow\quad \boxed{ds=\sqrt{(dx)^2+(dy)^2}}$$



* The Fundamental Theorem of Calculus, Part 1: If f is continuous on $[a,b]$, then the integral $g(x)=\int_a^x f(t)dt$ represents a function of x, which is continuous on $[a,b]$ and differentiable on (a,b), and the derivative of this function is $g'(x)=f(x)$.