§7.8 Improper Integrals

Type I: Integral over an Infinite Interval

a) $\int_a^t f(x)\,dx$ exists (as a finite number) for all $t \ge a$

$$\boxed{\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx}$$

provided that the limit exists (as a finite number).

b) $\int_t^b f(x)\,dx$ exists (as a finite number) for all $t \le b$

$$\boxed{\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx}$$

provided that the limit exists (as a finite number).

The improper integral $\int_a^{\infty} f(x)\,dx$ or $\int_{-\infty}^{b} f(x)\,dx$ is said to be

- CONVERGENT – if the limit exists (as a finite number)
- DIVERGENT – if the limit does not exist (as a finite number)

c) If both of the above defined improper integrals are convergent,

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{\infty} f(x)\,dx$$

Geometric Interpretation: When $f(x)$ is a positive function, each of these improper integrals represents the area under the curve $y = f(x)$, on the corresponding infinite interval.

Type II: Integral with a Discontinuous Integrand

a) If f is continuous on $[a,b)$ and <u>discontinuous</u> at b, then

$$\boxed{\int_a^{b} f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx}$$

provided that the limit exists (as a finite number).

b) If f is continuous on $(a,b]$ and <u>discontinuous</u> at a, then

$$\boxed{\int_a^{b} f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx}$$

provided that the limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is said to be

- CONVERGENT – if the limit exists (as a finite number)
- DIVERGENT – if the limit does not exist (as a finite number)

c) If f has a discontinuity at c, where $a < c < b$, and both of the above defined improper integrals are convergent, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Comparison Theorem

If f and g are two continuous functions such that

$$\boxed{0 \le g(x) \le f(x)}$$

for all $x \ge a$, then

a) If $\int_a^\infty f(x)\,dx$ is CONVERGENT, then $\int_a^\infty g(x)\,dx$ is CONVERGENT.

b) If $\int_a^\infty g(x)\,dx$ is DIVERGENT, then $\int_a^\infty f(x)\,dx$ is DIVERGENT.

Geometric interpretation:

The theorem holds if the integrals are considered as the areas under the curves.

- If the area under $y = f(x)$ (top curve) is FINITE, then the area under the bottom curve $y = g(x)$ is also FINITE.
- If the area under $y = g(x)$ (bottom curve) is INFINITE, then the area under the top curve $y = f(x)$ is also INFINITE.