§7.3 Trigonometric Substitution

- Integrals of the form $\int\sqrt{a^2-x^2}\,dx$, $\int\sqrt{a^2+x^2}\,dx$, and $\int\sqrt{x^2-a^2}\,dx$ cannot be solved with a simple substitution.
- We use an INVERSE SUBSTITUTION:

$$x = g(t) \quad \Rightarrow \quad dx = g'(t)\,dt \qquad \text{which yields} \qquad \int f(x)\,dx = \int f(g(t))\,g'(t)\,dt$$

Note: an inverse substitution can be made as long as g is a one-to-one function.

Integrand contains $\sqrt{a^2-x^2}$	Integrand contains $\sqrt{a^2+x^2}$	Integrand contains $\sqrt{x^2-a^2}$
Substitution: $x = a\sin\theta \;,\; \theta \in \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$	Substitution: $x = a\tan\theta \;,\; \theta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$	Substitution: $x = a\sec\theta \;,\; \theta \in \left[0,\frac{\pi}{2}\right) \cup \left[\pi,\frac{3\pi}{2}\right)$
Derivations: $x = a\sin\theta \Rightarrow dx = a\cos\theta\,d\theta$	Derivations: $x = a\tan\theta \Rightarrow dx = a\sec^2\theta\,d\theta$	Derivations: $x = a\sec\theta \Rightarrow dx = a\sec\theta\tan\theta\,d\theta$
$\sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta}$ $= \sqrt{a^2(1-\sin^2\theta)}$ $= a\sqrt{\cos^2\theta}$ $= a\lvert\cos\theta\rvert$ $= a\cdot\cos\theta$ as $\cos\theta \ge 0$	$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta}$ $= \sqrt{a^2(1+\tan^2\theta)}$ $= a\sqrt{\sec^2\theta}$ $= a\lvert\sec\theta\rvert$ $= a\sec\theta$ as $\sec\theta > 0$	$\sqrt{x^2-a^2} = \sqrt{a^2\sec^2\theta-a^2}$ $= \sqrt{a^2(\sec^2\theta-1)}$ $= a\sqrt{\tan^2\theta}$ $= a\lvert\tan\theta\rvert$ $= a\cdot\tan\theta$ as $\tan\theta \ge 0$
on the given interval.	on the given interval.	on the given interval.