Sec. 6.8 – L'Hospital's Rule

Let

- f and g are differentiable on some open interval I containing $x = a$
- $g'(x) \neq 0$ on the given interval when $x \neq a$ (with a possible exception at a)

Consider the limit:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)}} \qquad (*)$$

If

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = 0$ - then $(*)$ is an INDETERMINATE FORM of type $\frac{\mathbf{0}}{\mathbf{0}}$

or

$\lim_{x \to a} f(x) = \pm\infty$ and $\lim_{x \to a} g(x) = \pm\infty$ - then $(*)$ is an INDETERMINATE FORM of type $\frac{\infty}{\infty}$

Such limits can be evaluated using L'Hospital's Rule:

$$\boxed{\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}}$$

Therefore, the limit of the quotient of these functions equals the limit of the quotient of their DERIVATIVES.

Note: L'Hospital's Rule is also valid for limits with $x \to a^+$, $x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Indeterminate Product

Consider the limit:

$$\boxed{\lim_{x \to a} f(x) g(x)} \qquad (**)$$

If

$\lim_{x \to a} f(x) = 0$ and $\lim_{x \to a} g(x) = \infty$ - then $(**)$ is an INDETERMINATE FORM of type $\mathbf{0} \cdot \infty$.

To solve, write the product $f \cdot g$ as a quotient:

$$fg = \frac{f}{\frac{1}{g}} \qquad \text{or} \qquad fg = \frac{g}{\frac{1}{f}}$$

to obtain an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then apply L'Hospital's Rule.

Indeterminate Difference

Consider the limit:

$$\boxed{\lim_{x\to a}\left[\ f(x)-g(x)\ \right]} \qquad (***)$$

If

$\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$ - then $(***)$ is an INDETERMINATE FORM of type $\infty-\infty$

To solve, convert the difference $f-g$ into a quotient, to obtain an indeterminate form of type $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then apply L'Hospital's Rule.

Indeterminate Powers

Consider the limit:

$$\boxed{\lim_{x\to a}\left[\ f(x)\ \right]^{g(x)}} \qquad (****)$$

If

$\lim_{x\to a} f(x)=0$ and $\lim_{x\to a} g(x)=0$ - then $(****)$ is an INDETERMINATE FORM of type $\mathbf{0^0}$

$\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=0$ - then $(****)$ is an INDETERMINATE FORM of type $\mathbf{\infty^0}$

$\lim_{x\to a} f(x)=1$ and $\lim_{x\to a} g(x)=\infty$ - then $(****)$ is an INDETERMINATE FORM of type $\mathbf{1^\infty}$

To solve, take the natural logarithm of the indeterminate power and find the limit of the new expression.

$$y=f(x)^{g(x)} \quad\Rightarrow\quad \ln y=g(x)\cdot\ln\left[f(x)\right] \quad\Rightarrow\quad L=\lim_{x\to a}(\ln y)$$

The limit of the original expression is then obtained by using the exponential function:

$$\lim_{x\to a} y=e^L$$

Since $y=e^{\ln y}$, it follows that

$\ln y\to L \quad\Rightarrow\quad y=e^{\ln y}\to e^L$