§11.4 Comparison Tests

THE COMPARISON TEST

- If we have a series whose terms are *smaller* than those of a known *convergent* series, then our series is also convergent.
- If we have a series whose terms are *larger* than those of a known *divergent* series, then our series is also divergent.

Suppose that $\sum a_n$ and $\sum b_n$ are series with POSITIVE terms.

(i) If :
- $\sum b_n$ is <u>convergent</u>
- $\boxed{a_n \le b_n}$, for all n

then $\sum a_n$ is also CONVERGENT

(ii) If:
- $\sum b_n$ is <u>divergent</u>
- $\boxed{a_n \ge b_n}$, for all n

then $\sum a_n$ is also DIVERGENT

<u>Note</u>:

- The condition "for all n" may be relaxed to $n \ge N$ (since convergence of a series is not affected by a finite number of terms)
- The Comparison Test applies only if the terms of the given series are *smaller* than those of a *convergent* series or *larger* than those of a *divergent* series.

Suitable comparison series:

1. A p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$
2. A geometric series $\sum_{n=1}^{\infty} a\, r^{n-1}$ converges if $|r| < 1$ and diverges if $|r| \ge 1$

THE LIMIT COMPARISON TEST

Suppose that $\sum a_n$ and $\sum b_n$ are series with POSITIVE terms.

If $\boxed{\lim_{n\to\infty} \frac{a_n}{b_n} = c}$, for $0 < c < \infty$, then either
- both series CONVERGE
- both series DIVERGE

<u>Note</u>:

- Often a suitable comparison series $\sum b_n$ is obtained by keeping only the dominant terms in the numerator and the denominator (terms with highest powers)

ESTIMATING SUMS

If $a_n \le b_n$ and $\sum a_n$ converges by comparison with $\sum b_n$, we may be estimate the sum $\sum a_n$ by comparing remainders:

$$\boxed{R_n \le T_n}$$

$$R_n = s - s_n = a_{n+1} + a_{n+2} + \dots$$

$$T_n = t - t_n = b_{n+1} + b_{n+2} + \dots$$

where

1. $T_n = \sum_{n=i+1}^{\infty} ar^{i-1}$ - remainder of geometric series, can be calculated as an exact sum
2. $T_n \le \int_n^{\infty} \frac{1}{x^p}\,dx$ - remainder of p-series, estimated from the integral test