§11.3 Integral Test and Estimates of Sums

INTEGRAL TEST

Suppose that:

1. f is a continuous, positive, decreasing function on $[1, \infty)$
2. $a_n = f(n)$

Then the series $\sum_{n=1}^{\infty} a_n$ is convergent *iff* the improper integral $\int_1^{\infty} f(x)\,dx$ is convergent, that is,

a) If $\int_1^{\infty} f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

b) If $\int_1^{\infty} f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Note: in general $\sum_{n=1}^{\infty} a_n \neq \int_1^{\infty} f(x)\,dx$.

REMAINDER ESTIMATE for the Integral Test

Suppose that:

- f is a continuous, positive, decreasing function for $x \geq n$
- $a_n = f(n)$
- $\sum_{n=1}^{\infty} a_n$ is convergent

If $R_n = s - s_n$, then an error estimate of a partial sum approximation is

$$\int_{n+1}^{\infty} f(x)\,dx \leq R_n \leq \int_n^{\infty} f(x)\,dx$$

Adding s_n to each side of the inequality yields a more accurate approximation,

$$s_n + \int_{n+1}^{\infty} f(x)\,dx \quad \leq \quad s \quad \leq \quad s_n + \int_n^{\infty} f(x)\,dx$$

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An important example:

The p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$