§11.2 Series

INFINITE SERIES (series)	- THE SUM of the terms of an infinite sequence $\{a_n\}_{n=1}^{\infty}$ Notation: $\sum_{n=1}^{\infty} a_n$ or $\sum a_n$
PARTIAL SUM	$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$ - the SUM of the first *n*-terms of a series
$\{s_n\}$	- sequence of partial sums $s_1 = \sum_{k=1}^{1} a_k = a_1$ $s_2 = \sum_{k=1}^{2} a_k = a_1 + a_2$ $s_3 = \sum_{k=1}^{3} a_k = a_1 + a_2 + a_3$ $\vdots$ $s_n = \sum_{k=1}^{n} a_n = a_1 + a_2 + a_3 + \cdots + a_n$ $\vdots$ Given: 1. series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + ...$ 2. sequence of partial sums $s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n$ If the sequence $\{s_n\}$ is convergent and $\lim_{n\to\infty} s_n = s$, then $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s$
SUM of a series	- THE LIMIT OF THE SEQUENCE OF PARTIAL SUMS: $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} s_n = s$
CONVERGENT series	- a series whose sequence of partial sums $\{s_n\}$ CONVERGES
DIVERGENT series	- a series whose sequence of partial sums $\{s_n\}$ DIVERGES
THEOREM	If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.
TEST for DIVERGENCE	If $\lim_{n\to\infty} a_n = \begin{cases} \neq 0 \\ DNE \end{cases}$, then the series $\sum_{n=1}^{\infty} a_n$ is DIVERGENT

Note: with a series $\sum a_n$, we associate two sequences:

1. $\{s_n\}$ - sequence of partial sums
2. $\{a_n\}$ - sequence of the terms of the series

If a series is convergent, then the sum is the limit of the sequence of partial sums and also $\lim_{n\to\infty} a_n = 0$.

The converse does not hold: if $\lim_{n\to\infty} a_n = 0$ we cannot conclude anything about the convergence of the series

However, if we find $\lim_{n\to\infty} a_n \neq 0$, then the series is divergent.

Some Important series

GEOMETRIC series	$\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$, where $a \neq 0$ - an infinite sequence in which each next term is obtained from the preceding term by multiplying it by r r - COMMON RATIO Sum of the geometric series: The geometric series $\sum_{n=1}^{\infty} a \cdot r^{n-1} = a + ar + ar^2 + ar^3 + \ldots + ar^n + \ldots$ is: 1. Convergent if $\lvert r \rvert < 1$, in which case the sum of the series is $\sum_{n=1}^{\infty} a \cdot r^{n-1} = \frac{a}{1-r}$ 2. Divergent whenever $\lvert r \rvert \geq 1$
HARMONIC series	$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots$ - diverges
SUM OF INTEGERS	$\sum_{n=1}^{\infty} n = 1 + 2 + 3 + \ldots + n + \ldots$ - diverges $s_n = \sum_{i=1}^{n} i = 1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$ - sum of the first *n* integers

Properties of Convergent Series

Theorem:

If $\sum a_n$ and $\sum b_n$ are two convergent series, then so are the series:

(i) $\sum ca_n$, where c is a constant $\qquad \sum_{n=1}^{\infty} ca_n = c\sum_{n=1}^{\infty} a_n$

(ii) $\sum (a_n + b_n)$ $\qquad \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

(iii) $\sum (a_n - b_n)$ $\qquad \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Note: A finite number of terms do not affect the convergence or divergence of a series.