§11.1 Sequences

CONCEPT	EXPLANATION
SEQUENCE	**– a list of numbers written in a definite order** $\{a_1, a_2, a_3, \ldots, a_n, \ldots\}$ $\{a_n\}$ $\{a_n\}_{n=1}^{\infty}$
Term	– one of the numbers in a sequence a_1 - 1st term a_2 - 2nd term $\vdots$ a_n - *n*th or general term
Graph of a sequence	- isolated points with coordinates $(1, a_1), (2, a_2), (3, a_3), \ldots (n, a_n), \ldots$ - a sequence can also be graphed on a real line
Limit of a sequence	$\boxed{\lim_{n\to\infty} a_n = L}$ - if terms a_n can get as close to L as we like by taking n sufficiently large $\forall \varepsilon > 0 \quad \exists N > 0 \quad s.t. \quad \boxed{n > N \ \Rightarrow \ \lvert a_n - L \rvert < \varepsilon}$ - more precise definition For any positive value ε there exists a positive integer N such that all terms with indices $n > N$ are within the ε-distance of L **Convergent** sequence: $\lim_{n\to\infty} a_n$ exists **Divergent** sequence: $\lim_{n\to\infty} a_n$ is infinite or DNE ▫ sequence that diverges to infinity: $\lim_{n\to\infty} a_n = \infty$ $\forall M > 0 \quad \exists N \quad s.t. \quad n > N \ \Rightarrow \ a_n > M$ For any positive number M there exists a positive integer N such that all terms with indices $n > N$ are above M. **Limit Laws** for Sequences 1. $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ 2. $\lim_{n\to\infty}(a_n - b_n) = \lim_{n\to\infty} a_n - \lim_{n\to\infty} b_n$ 3. $\lim_{n\to\infty} c \cdot a_n = c \lim_{n\to\infty} a_n$ 4. $\lim_{n\to\infty}(a_n \cdot b_n) = \lim_{n\to\infty} a_n \cdot \lim_{n\to\infty} b_n$ 5. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to\infty} a_n}{\lim_{n\to\infty} b_n}$, if $\lim_{n\to\infty} b_n \neq 0$ 6. $\lim_{n\to\infty} a_n{}^p = \left(\lim_{n\to\infty} a_n\right)^p$, if $p > 0$, $a_n > 0$

Theorems about limits of sequences:

Theorem 1	If $f(n) = a_n$ for integer *n*, and $\lim_{x\to\infty} f(x) = L$, then $\boxed{\lim_{n\to\infty} a_n = L}$ *Note*: since the only difference between $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} f(x) = L$ is that *n* is an integer, we have $\lim_{n\to\infty} a_n = L$. Applications: 1. $\boxed{\lim_{n\to\infty} \frac{1}{n^r} = 0}$, since $\lim_{x\to\infty} \frac{1}{x^r} = 0$. 2. L'Hospital's Rule cannot be applied directly to sequences, only to a function of a real variable. .
Theorem 2	If $\lim_{n\to\infty} a_n = L$ and the function f is continuous at L, then $\boxed{\lim_{n\to\infty} f(a_n) = f(L)}$.
Squeeze Theorem	If $a_n \le b_n \le c_n$ for $n \ge n_0$ and $\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} c_n$, then $\boxed{\lim_{n\to\infty} b_n = L}$.
Theorem 3	If $\lim_{n\to\infty} \lvert a_n \rvert = 0$, then $\lim_{n\to\infty} a_n = 0$.

Geometric sequence

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & , \quad \lvert r \rvert < 1 \\ 1 & , \quad r = 1 \\ diverges & , \quad otherwise \end{cases}$$

$$\lim_{n\to\infty} r^n = \begin{cases} 0 & , \ \lvert r \rvert < 1 \quad e.g. \ \left(\tfrac{1}{2}\right)^n \to 0 \ , \ \text{as} \ n \to \infty \\ 1 & , \ r = 1 \quad \text{e.g.} \ 1^n \to 1 \ , \ \text{as} \ n \to \infty \\ \infty & , \ r > 1 \quad \text{e.g.} \ 2^n \to \infty \ , \ \text{as} \ n \to \infty \\ DNE & , \ r < -1 \quad \text{e.g.} \lim_{n\to\infty} (-2)^n \text{ oscillates as } n \to \infty \\ DNE & , \ r = -1 \quad e.g. \lim_{n\to\infty} (-1)^n \text{ oscillates as } n \to \infty \end{cases}$$

Monotonic and Bounded Sequences

Monotonic sequence	- **either increasing or decreasing** Increasing sequence: $\boxed{a_{n+1} > a_n}$, for all $n \ge 1$ Decreasing sequence: $\boxed{a_{n+1} < a_n}$, for all $n \ge 1$
Bounded sequence	- **bounded from both above and below** Bounded above: $a_n \le M$ Bounded below: $a_n \ge M$ **Least upper bound**: $\boxed{b \le M}$ an upper bound b that it is smaller than any other upper bound M Note: a) A sequence can be bounded above but not below. b) Not every bounded sequence is convergent.
Theorem	**EVERY BOUNDED, MONOTONIC SEQUENCE IS CONVERGENT.**
Axiom	If S is a nonempty set of real numbers with upper bound M, then S has a least upper bound b.