§10.2 Calculus with Parametric Curves

TANGENT LINE at a point on a parametric curve

Suppose a curve C is given by the parametric equations:

$$\boxed{x = f(t) \quad , \quad y = g(t)}$$

where f and g are differentiable functions. To determine the tangent line at a point on the curve where y is also a differentiable function of x, use the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

> This equation allows us to find the slope *dy/dx* of the tangent to a parametric curve without having to eliminate the parameter

Then

$$\boxed{\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \quad , \quad \text{if} \quad \frac{dx}{dt} \neq 0}$$

If

- $\frac{dy}{dt} = 0$, the curve has a **horizontal** tangent (provided that $dx/dt = 0$)
- $\frac{dx}{dt} = 0$, the curve has a **vertical** tangent (provided that $dy/dt = 0$)

To find the second derivative, differentiate the first derivative:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} \quad \text{or} \quad \boxed{y'' = \frac{\frac{d}{dt}(y')}{\frac{dx}{dt}}}$$

> Easy way to remember: replace *y* in the expression for the first derivative by *y′*.

AREA OF THE REGION *under* a parametric curve or *enclosed* by a parametric curve

If the curve is traced out <u>once</u> by the parametric equations $x = f(t)$ and $y = g(t)$, where $\alpha \le t \le \beta$, the area below or enclosed by the parametric curve can be calculated by using the *Substitution Rule* for Definite Integrals:

$$A = \int_a^b y\,dx = \boxed{\int_\alpha^\beta g(t) f'(t)\,dt} \quad , \quad \text{or} \quad A = \int_\beta^\alpha g(t) f'(t)\,dt$$

Note: The area is obtained by moving in the direction from left to right along the x-axis, which is not necessarily the direction of the increase for the parameter t.

ARC LENGTH of a parametric curve

Recall the formula for the length L of a curve C given by $y = F(x)$, $a \le x \le b$ where F is a continuous function:

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

Suppose that a curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$, where $dx/dt = f'(t) > 0$. This means that C is traversed once, from left to right, as t increases from α to β; also $f(\alpha) = a$ and $f(\beta) = b$. Then, by the Substitution Rule:

$$L = \int_a^b \sqrt{1 + \left(\tfrac{dy}{dx}\right)^2}\, dx = \int_\alpha^\beta \sqrt{1 + \left(\frac{dy/dt}{dx/dt}\right)^2}\, \frac{dx}{dt} dt = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

since $dx/dt = 0$.

Theorem:

If

- Curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$
- Functions f' and g' are continuous on $[\alpha, \beta]$
- C is traversed exactly once as t increases from α to β

then the length of C is

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

SURFACE AREA of an object generated by rotating a parametric curve about an axis

Suppose that

- Curve C is described by the parametric equations $x = f(t)$ and $y = g(t)$, $\alpha \le t \le \beta$
- Functions f' and g' are continuous on $[\alpha, \beta]$
- $g(t) \ge 0$

then the area of a surface obtained by rotating C about the x-axis is

$$S = \int_\alpha^\beta 2\pi y(t) \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$