§10.1 Curves Defined by Parametric Equations

<table>
<tr>
<td>PARAMETRIC EQUATIONS</td>
<td>
Let x and y are expressed as functions of a third variable t, called a *parameter*:

$$\boxed{x = f(t) \quad , \quad y = g(t)}$$ - parametric equations

As t varies, the point $(x, y) = (f(t), g(t))$ varies and traces a *parametric curve* C.

Note: if the parameter is restricted to a certain range $a \le t \le b$, then:

$(f(a), g(a))$ - initial point

$(f(b), g(b))$ - terminal point
</td>
</tr>
<tr>
<td>Ellipse</td>
<td>
▫ The parametric equation of an ellipse centered at (x_0, y_0) with major semi-axis of length a and minor semi-axis of length b are:

For counterclockwise motion:

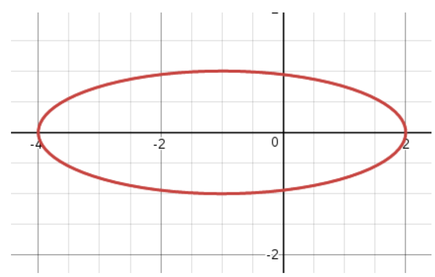
$$x = x_0 + a\cos t$$
$$y = y_0 + b\sin t$$

For clockwise motion:

$$x = x_0 + a\sin t$$
$$y = y_0 + b\cos t$$

Example: Ellipse centered at (-1, 0), with $a = 3$ and $b = 1$

$$x = -1 + 3\cos t$$
$$y = \sin t$$
$$0 \le t \le 2\pi$$



</td>
</tr>
<tr>
<td>Graphing Devices</td>
<td>
▫ Most graphing calculators and computer graphing programs can graph curves defined by parametric equations. Watch a parametric curve as it is drawn by a graphing calculator because the points are *plotted in order* as the parameter values increase.

▫ Desmos online graphing calculator: www.desmos.com/calculator
</td>
</tr>
<tr>
<td>Applications</td>
<td>
▫ Bezier curves (special parametric curves) used in computer-aided design (CAD) for

– manufacturing

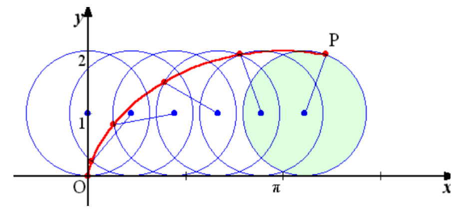
– specifying the shapes of symbols in laser printers
</td>
</tr>
</table>

Cycloid:

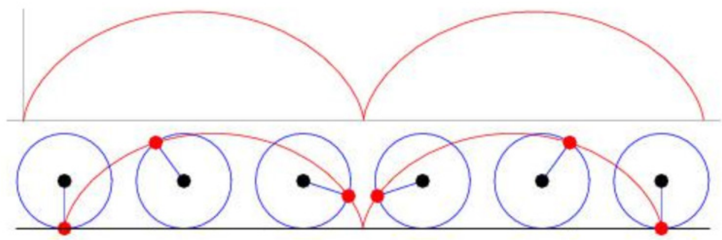
Curve traced by a point on the circumference of a circle as the circle rolls along a straight line.

$$x = r\theta - r\sin\theta = r(\theta - \sin\theta)$$

$$y = r - r\cos\theta = r(1 - \cos\theta)$$



One arch of the cycloid comes from one rotation of the circle, so $0 \le \theta \le 2\pi$.



Historical remarks:

The cycloid is the solution to the ***brachistochrone* problem** (path of quickest descend): Find the curve along which a particle slides in shortest time (under the influence of gravity) from a point *A* to a lower point *B* not directly beneath *A*. Solved by Johan Bernoulli (1696), as well as by Jacob Bernoulli, Gottfried Leibniz, Isaac Newton, and Marquis de L'Hospital.

The cycloid is also the solution to ***tautochrone* problem**: Find the curve along which a particle will slide to the curve's lowest point in equal time regardless of the particle's initial location along it. Christiaan Huygens (1673) proposed that pendulum clocks swing in cycloidal arcs because then it takes the same time to make a complete oscillation whether it swings through a wide or a small arc.

Families of Parametric Curves

Conchoids of Nicomedes (200 BC): family of curves with parametric equations:

$$x = a + \cos t\,, \ \ y = a\tan t + \sin t$$

All curves (but $a = 0$) have two branches, approaching the vertical asymptote $x = a$ as $x \to a$.

$a < -1$	Two smooth branches
$a = -1$	Right branch gets a cusp
$-1 < a < 0$	Cusp turns into a growing loop
$a = 0$	The loop forms a circle
$0 < a < 1$	Left branch has a loop
$a = 1$	Left branch gets a cusp
$a > 1$	Smooth branches